Algebraic independence of the values of certain functions at distinct algebraic points(Analytic Number Theory and Surrounding Areas)

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数理解析研究所講究録
1511巻 2006年 10-18

Algebraic independence of the values of certain functions at distinct algebraic points

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1 Introduction and results

Loxton and van der Poorten [2] obtained the following result: Let

$$F(z)=\sum_{k=0}^{\infty} z^{d^{k}},$$

where d is an integer greater than 1, and let $\alpha_1, \cdots, \alpha_r$ be algebraic numbers with $0<|\alpha_i|<1$ $(1 \leq i \leq r)$. Then the following three properties are equivalent:

(i) $F(\alpha_1), \ldots, F(\alpha_r)$ are algebraically dependent.

(ii) $1, F(\alpha_1), \ldots, F(\alpha_r)$ are linearly dependent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

(iii) There exist a non-empty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_t}\}$ of $\{\alpha_1, \ldots, \alpha_r\}$, nonnegative integers $k_1, \ldots, k_t$, roots of unity $\zeta_1, \ldots, \zeta_t$, an algebraic number γ with $\alpha_{i_l}^{d^{k_l}}=\zeta_l \gamma$ $(1 \leq l \leq t)$, and algebraic numbers $\xi_1, \ldots, \xi_t$, not all zero, such that

$$\sum_{l=1}^{t} \xi_l \zeta_l^{d^{k}}=0 \quad (k=0,1,2, \ldots).$$

In contrast with this result we consider the power series

$$f(z)=\sum_{k=0}^{\infty} z^{a_k},$$

where $\{a_k\}_{k \geq 0}$ is a linear recurrence of positive integers which is not a geometric progression and which satisfies

$$a_{k+n}=c_1 a_{k+n-1}+\cdots+c_n a_k \quad (k=0,1,2, \ldots), \tag{1}$$

where $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. We assume that the polynomial $\Phi(X)=X^n-c_1 X^{n-1}-\cdots-c_n$ associated with (1) satisfies $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. For this power series $f(z)$, the author obtained the necessary and sufficient condition for the numbers $f(\alpha_1), \ldots, f(\alpha_r)$ to be algebraically dependent.

DEFINITION 1. We say that the algebraic numbers $\alpha_1, \ldots, \alpha_r$ with $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$ are *$\{a_k\}_{k\geq 0}$-dependent* if there exist a non-empty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_t}\}$ of $\{\alpha_1, \ldots, \alpha_r\}$, roots of unity $\zeta_1, \ldots, \zeta_t$, an algebraic number γ with $\alpha_{i_l} = \zeta_l \gamma$ $(1 \leq l \leq t)$, and algebraic numbers $\xi_1, \ldots, \xi_t$, not all zero, such that

$$\sum_{l=1}^{t} \xi_l \zeta_l^{a_k} = 0$$

for all sufficiently large k.

Theorem 1 (A special case of Theorem 2 in [6]). *Let $\{a_k\}_{k\geq 0}$ be a linear recurrence satisfying* (1). *Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$. Then the following three properties are equivalent:*

(i) *$f(\alpha_1), \ldots, f(\alpha_r)$ are algebraically dependent.*

(ii) *$1, f(\alpha_1), \ldots, f(\alpha_r)$ are linearly dependent over $\overline{\mathbb{Q}}$.*

(iii) *$\alpha_1, \ldots, \alpha_r$ are $\{a_k\}_{k\geq 0}$-dependent.*

REMARK 1. In Theorem 1 it is obvious that the property (iii) implies (ii), since $\sum_{l=1}^{t} \xi_l f(\alpha_{i_l}) \in \overline{\mathbb{Q}}$ if $\alpha_1, \ldots, \alpha_r$ are $\{a_k\}_{k\geq 0}$-dependent.

REMARK 2. As a special case of the result of Nishioka [4], the three properties (i)–(iii) in Theorem 1 are equivalent also for a gap series $\sum_{k=0}^{\infty} z^{a_k}$ with $\{a_k\}_{k\geq 0}$ an increasing sequence of positive integers such that $\lim_{k\to\infty} a_{k+1}/a_k = \infty$. In the case of our linear recurrence $\{a_k\}_{k\geq 0}$ satisfying (1), we have $\lim_{k\to\infty} a_{k+1}/a_k = \rho$ with $1 < \rho < \infty$.

In what follows, let

$$f(z) = \sum_{k=0}^{\infty} z^{a_k}, \qquad g(z) = \sum_{k=0}^{\infty} \frac{z^{a_k}}{1 - z^{a_k}}, \qquad h(z) = \prod_{k=0}^{\infty} (1 - z^{a_k}).$$

The author proved the following:

Theorem 2 ([7, Theorem 5]). *Let $\{a_k\}_{k\geq 0}$ be a linear recurrence satisfying* (1). *Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$ such that none of α_i/α_j $(1 \leq i < j \leq r)$ is a root of unity. Then the $3r$ numbers $f(\alpha_i)$, $g(\alpha_i)$, $h(\alpha_i)$ $(1 \leq i \leq r)$ are algebraically independent.*

REMARK 3. If $\{a_k\}_{k\geq 0}$ is a geometric progression, namely $a_k = ad^k$ $(k \geq 0)$ for some integers $a \geq 1$ and $d \geq 2$, each of the $3r$ numbers in Theorem 2 is transcendental by the theorem of Mahler [3] ; however Theorem 2 is not valid in this case, since there exist the following relations over $\overline{\mathbb{Q}}$: Let

$$F(z)=\sum_{k=0}^{\infty} z^{ad^k}, \qquad G(z)=\sum_{k=0}^{\infty} \frac{z^{ad^k}}{1-z^{ad^k}}, \qquad H(z)=\prod_{k=0}^{\infty}(1-z^{ad^k}),$$

and let α be an algebraic number with $0<|\alpha|<1$. Then

$$F(\alpha)-F(\alpha^d)=\alpha^a, \qquad G(\alpha)-G(\alpha^d)=\frac{\alpha^a}{1-\alpha^a}, \qquad \frac{H(\alpha)}{H(\alpha^d)}=1-\alpha^a,$$

whereas α/α^d is not a root of unity.

Remark 4. The assumption in Theorem 2 that none of α_i/α_j $(1\le i<j\le r)$ is a root of unity cannot be removed. For example, suppose that the initial values $a_0,\ldots,a_{n-1}$ are divided by an integer $d>1$. Then by the linear recurrence relation (1), a_k is divided by d for any $k\ge 0$. If α_i/α_j is a d-th root of unity for some distinct i and j, then $\alpha_i^{a_k}=\alpha_j^{a_k}$ $(k\ge 0)$ and so the numbers considered in Theorem 2 are algebraically dependent. Even in some cases where $a_0,\ldots,a_{n-1}$ have no common factor, the assumption is also inevitable as shown in the following example:

Let $\{a_k\}_{k\ge 0}$ be a linear recurrence defined by

$$a_0=2, \quad a_1=3, \quad a_{k+2}=6a_{k+1}+a_k \quad (k=0,1,2,\ldots).$$

We put

$$f(z)=\sum_{k=0}^{\infty} z^{a_k}, \qquad g(z)=\sum_{k=0}^{\infty}\frac{z^{a_k}}{1-z^{a_k}}, \qquad h(z)=\prod_{k=0}^{\infty}(1-z^{a_k}).$$

Let α be an algebraic number with $0<|\alpha|<1$ and let $\zeta=e^{\pi\sqrt{-1}/3}=(1+\sqrt{-3})/2$. Then

$$2f(\alpha)+f(\zeta\alpha)-f(\zeta^2\alpha)-2f(\zeta^3\alpha)-f(\zeta^4\alpha)+f(\zeta^5\alpha)=0,$$
$$2g(\alpha)+g(\zeta\alpha)-g(\zeta^2\alpha)-2g(\zeta^3\alpha)-g(\zeta^4\alpha)+g(\zeta^5\alpha)=0,$$

and

$$h(\alpha)^2h(\zeta\alpha)h(\zeta^2\alpha)^{-1}h(\zeta^3\alpha)^{-2}h(\zeta^4\alpha)^{-1}h(\zeta^5\alpha)=1,$$

since $a_{2k}\equiv 2 \pmod 6$ and $a_{2k+1}\equiv 3 \pmod 6$ for any $k\ge 0$.

The author obtained the necessary and sufficient condition for the $3r$ numbers $f(\alpha_i),\ g(\alpha_i),\ h(\alpha_i)$ $(1\le i\le r)$ in Theorem 2 to be algebraically dependent:

Theorem 3 ([8]). *Let $\{a_k\}_{k\ge 0}$ be a linear recurrence satisfying (1). Let $\alpha_1,\ldots,\alpha_r$ be algebraic numbers with $0<|\alpha_i|<1$ $(1\le i\le r)$. Then the numbers $f(\alpha_i),\ g(\alpha_i),\ h(\alpha_i)$ $(1\le i\le r)$ are algebraically dependent if and only if the algebraic numbers $\alpha_1,\ldots,\alpha_r$ are $\{a_k\}_{k\ge 0}$-dependent.*

Combining Theorems 1 and 3, we immediately have the following:

Theorem 4 ([8]). *Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$. If the numbers $f(\alpha_1), \ldots, f(\alpha_r)$ are algebraically independent, then so are the numbers $f(\alpha_i),\ g(\alpha_i),\ h(\alpha_i)$ $(1 \le i \le r)$.*

Theorem 4 implies the following:

Theorem 5 ([8]). *Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$. Then*

$$\begin{aligned} &\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha_1), \ldots, f(\alpha_r), g(\alpha_1), \ldots, g(\alpha_r), h(\alpha_1), \ldots, h(\alpha_r)\Big) \\ \ge\ & 3 \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha_1), \ldots, f(\alpha_r)\Big). \end{aligned} \tag{2}$$

The following is an example in which the equality of (2) holds:

EXAMPLE 1. Let $\{a_k\}_{k\ge0}$ be a linear recurrence defined by

$$a_0 = 1, \quad a_1 = 2, \quad a_{k+2} = 3a_{k+1} + a_k \quad (k = 0, 1, 2, \ldots).$$

We put

$$f(z) = \sum_{k=0}^{\infty} z^{a_k}, \qquad g(z) = \sum_{k=0}^{\infty} \frac{z^{a_k}}{1 - z^{a_k}}, \qquad h(z) = \prod_{k=0}^{\infty} (1 - z^{a_k}).$$

Let α be an algebraic number with $0 < |\alpha| < 1$ and let $\omega = e^{2\pi\sqrt{-1}/3} = (-1 + \sqrt{-3})/2$. Since $a_{2k} \equiv 1 \pmod 3$ and $a_{2k+1} \equiv 2 \pmod 3$ for any $k \ge 0$, the numbers α, $\omega\alpha$, and α^3 are not $\{a_k\}_{k\ge0}$-dependent. Therefore the numbers $f(\alpha), f(\omega\alpha), f(\alpha^3), g(\alpha), g(\omega\alpha), g(\alpha^3), h(\alpha), h(\omega\alpha), h(\alpha^3)$ are algebraically independent by Theorem 3. Noting that $f(\alpha) + f(\omega\alpha) + f(\omega^2\alpha) = 0$, $g(\alpha) + g(\omega\alpha) + g(\omega^2\alpha) = 3g(\alpha^3)$, and $h(\alpha)h(\omega\alpha)h(\omega^2\alpha) = h(\alpha^3)$, we see that

$$\begin{aligned} \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3)\Big) &= 3, \\ \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(g(\alpha), g(\omega\alpha), g(\omega^2\alpha), g(\alpha^3)\Big) &= 3, \\ \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(h(\alpha), h(\omega\alpha), h(\omega^2\alpha), h(\alpha^3)\Big) &= 3, \end{aligned}$$

and

$$\begin{aligned} &\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3), \\ &g(\alpha), g(\omega\alpha), g(\omega^2\alpha), g(\alpha^3), h(\alpha), h(\omega\alpha), h(\omega^2\alpha), h(\alpha^3)\Big) = 9. \end{aligned}$$

As shown in Remark 4 or in the example above, it seems complicated to state the necessary and sufficient condition for the values of the Lambert series $g(z)$ and the infinite product $h(z)$ at $\{a_k\}_{k\ge0}$-dependent algebraic numbers $\alpha_1, \ldots, \alpha_r$ to be algebraically independent. In Theorem 6 below we establish an easily confirmable condition under which such values are algebraically independent.

DEFINITION 2. We say that the algebraic numbers $\alpha_1, \ldots, \alpha_r$ with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$ are *strongly* $\{a_k\}_{k\ge0}$*-dependent* if there exist a non-empty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_t}\}$ of $\{\alpha_1, \ldots, \alpha_r\}$, N-th roots of unity $\zeta_1, \ldots, \zeta_t$, an algebraic number γ with $\alpha_{i_l} = \zeta_l\gamma$ $(1 \le l \le t)$, and algebraic numbers $\xi_1, \ldots, \xi_t$, not all zero, such that

$$\sum_{l=1}^{t} \xi_l \zeta_l^{ma_k} = 0, \qquad m = 1, \ldots, N-1, \qquad \text{g.c.d.}(m, N) = 1,$$

for all sufficiently large k.

It is clear that, if the algebraic numbers $\alpha_1, \ldots, \alpha_r$ with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$ are strongly $\{a_k\}_{k\ge0}$-dependent, then they are $\{a_k\}_{k\ge0}$-dependent.

The following theorem is more precise than Theorem 4 above.

Theorem 6 ([8]). *Let $\{a_k\}_{k\ge0}$ be a linear recurrence satisfying (1). Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \le i \le r)$. Suppose that the algebraic numbers $\alpha_1, \ldots, \alpha_r$ are not strongly $\{a_k\}_{k\ge0}$-dependent. Assume further that $\alpha_1, \ldots, \alpha_\rho$ $(\rho \le r)$ are not $\{a_k\}_{k\ge0}$-dependent or equivalently that the numbers $f(\alpha_1), \ldots, f(\alpha_\rho)$ are algebraically independent. Then the numbers $f(\alpha_1), \ldots, f(\alpha_\rho), g(\alpha_1), \ldots, g(\alpha_r), h(\alpha_1), \ldots, h(\alpha_r)$ are algebraically independent.*

Using Theorem 6, we have an example in which the strict inequality of (2) holds:

EXAMPLE 2. Let $\{a_k\}_{k\ge0}$ be a linear recurrence defined by

$$a_0 = 1, \quad a_1 = 3, \quad a_{k+2} = 3a_{k+1} + a_k \quad (k = 0, 1, 2, \ldots).$$

We put

$$f(z) = \sum_{k=0}^{\infty} z^{a_k}, \qquad g(z) = \sum_{k=0}^{\infty} \frac{z^{a_k}}{1 - z^{a_k}}, \qquad h(z) = \prod_{k=0}^{\infty}(1 - z^{a_k}).$$

Let α be an algebraic number with $0 < |\alpha| < 1$ and let $\omega = e^{2\pi\sqrt{-1}/3} = (-1 + \sqrt{-3})/2$. Since $a_{2k} \equiv 1 \pmod 3$ and $a_{2k+1} \equiv 0 \pmod 3$ for any $k \ge 0$, the numbers α, $\omega\alpha$, $\omega^2\alpha$, and α^3 are not strongly $\{a_k\}_{k\ge0}$-dependent and the numbers α, $\omega\alpha$, and α^3 are not $\{a_k\}_{k\ge0}$-dependent. Therefore the numbers $f(\alpha), f(\omega\alpha), f(\alpha^3), g(\alpha), g(\omega\alpha), g(\omega^2\alpha), g(\alpha^3), h(\alpha), h(\omega\alpha), h(\omega^2\alpha), h(\alpha^3)$ are algebraically independent by Theorem 6 with $\rho = 3$ and $r = 4$. Noting that $\omega f(\alpha) - (\omega + 1)f(\omega\alpha) + f(\omega^2\alpha) = 0$, we see that

$$\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3)\Big) = 3,$$

$$\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3),$$
$$g(\alpha), g(\omega\alpha), g(\omega^2\alpha), g(\alpha^3), h(\alpha), h(\omega\alpha), h(\omega^2\alpha), h(\alpha^3)\Big) = 11,$$

and so

$$\begin{aligned} &\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3), \\ &g(\alpha), g(\omega\alpha), g(\omega^2\alpha), g(\alpha^3), h(\alpha), h(\omega\alpha), h(\omega^2\alpha), h(\alpha^3)\Big) \\ > \;& 3\,\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}\Big(f(\alpha), f(\omega\alpha), f(\omega^2\alpha), f(\alpha^3)\Big). \end{aligned}$$

2 Proof of Theorems 3 and 6

Proof of Theorem 3. If the algebraic numbers $\alpha_1, \dots, \alpha_r$ are $\{a_k\}_{k\geq 0}$-dependent, then the numbers $f(\alpha_i)$, $g(\alpha_i)$, $h(\alpha_i)$ $(1 \leq i \leq r)$ are algebraically dependent, since so are the numbers $f(\alpha_1), \dots, f(\alpha_r)$ by Theorem 1 with Remark 1. Conversely, if the algebraic numbers $\alpha_1, \dots, \alpha_r$ are not $\{a_k\}_{k\geq 0}$-dependent, then by Theorem 6 with $\rho = r$ the numbers $f(\alpha_i)$, $g(\alpha_i)$, $h(\alpha_i)$ $(1 \leq i \leq r)$ are algebraically independent. This completes the proof of the theorem.

Sketch of the proof of Theorem 6. Suppose on the contrary that the numbers $f(\alpha_1), \dots, f(\alpha_\rho), g(\alpha_1), \dots, g(\alpha_r), h(\alpha_1), \dots, h(\alpha_r)$ are algebraically dependent. There exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_s$ with $0 < |\beta_j| < 1$ $(1 \leq j \leq s)$ such that

$$\alpha_i = \zeta_i \prod_{j=1}^{s} \beta_j^{\,e_{ij}} \quad (1 \leq i \leq r), \tag{3}$$

where $\zeta_1, \dots, \zeta_r$ are roots of unity and e_{ij} $(1 \leq i \leq r,\ 1 \leq j \leq s)$ are nonnegative integers (cf. Nishioka [5, Lemma 3.4.9]). Take a positive integer N such that $\zeta_i^N = 1$ for any i $(1 \leq i \leq r)$. We can choose a positive integer p and a nonnegative integer q such that $a_{k+p} \equiv a_k \pmod N$ for any $k \geq q$. Let y_{jl} $(1 \leq j \leq s,\ 1 \leq l \leq n)$ be variables and let $\boldsymbol{y} = (y_{11}, \dots, y_{1n}, \dots, y_{s1}, \dots, y_{sn})$. Define the auxiliary functions

$$\begin{aligned} f_i(\boldsymbol{y}) &= \sum_{k=q}^{\infty} \zeta_i^{a_k} \prod_{j=1}^{s} (y_{j1}^{a_{k+n-1}} \cdots y_{jn}^{a_k})^{e_{ij}} \qquad (1 \leq i \leq \rho), \\ g_i(\boldsymbol{y}) &= \sum_{k=q}^{\infty} \frac{\zeta_i^{a_k} \prod_{j=1}^{s} (y_{j1}^{a_{k+n-1}} \cdots y_{jn}^{a_k})^{e_{ij}}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} (y_{j1}^{a_{k+n-1}} \cdots y_{jn}^{a_k})^{e_{ij}}} \qquad (1 \leq i \leq r), \end{aligned}$$

and

$$h_i(\boldsymbol{y}) = \prod_{k=q}^{\infty} \Big(1 - \zeta_i^{a_k} \prod_{j=1}^{s} (y_{j1}^{a_{k+n-1}} \cdots y_{jn}^{a_k})^{e_{ij}}\Big) \qquad (1 \leq i \leq r).$$

Letting

$$\beta = (\underbrace{1,\ldots,1}_{n-1},\beta_1,\ldots\ldots,\underbrace{1,\ldots,1}_{n-1},\beta_s),$$

we see by (3) that

$$f_i(\beta)=\sum_{k=q}^{\infty}\alpha_i^{a_k},\qquad g_i(\beta)=\sum_{k=q}^{\infty}\frac{\alpha_i^{a_k}}{1-\alpha_i^{a_k}},\qquad h_i(\beta)=\prod_{k=q}^{\infty}(1-\alpha_i^{a_k}).$$

Hence the values $f_1(\beta),\ldots,f_\rho(\beta),g_1(\beta),\ldots,g_r(\beta),h_1(\beta),\ldots,h_r(\beta)$ are algebraically dependent. Let Ω be a multiplicative transformation for the variables $y_{11},\ldots,y_{1n},\ldots,y_{s1},\ldots,y_{sn}$ sending $y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k}$ to $y_{j1}^{a_{k+p+n-1}}\cdots y_{jn}^{a_{k+p}}$ for $j=1,\ldots,s$. Then $f_1(\boldsymbol{y}),\ldots,f_\rho(\boldsymbol{y}),g_1(\boldsymbol{y}),\ldots,g_r(\boldsymbol{y}),h_1(\boldsymbol{y}),\ldots,h_r(\boldsymbol{y})$ satisfy the functional equations

$$\begin{aligned} f_i(\boldsymbol{y}) &= f_i(\Omega\boldsymbol{y})+\sum_{k=q}^{p+q-1}\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}},\\ g_i(\boldsymbol{y}) &= g_i(\Omega\boldsymbol{y})+\sum_{k=q}^{p+q-1}\frac{\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}}{1-\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}}, \end{aligned}$$

and

$$h_i(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1}\left(1-\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}\right)\right)h_i(\Omega\boldsymbol{y}),$$

since $a_{k+p}\equiv a_k \pmod N$ for any $k\ge q$. By Mahler's method improved by Kubota [1], at least one of the following two cases arises:

(i) There are algebraic numbers $b_1,\ldots,b_\rho,c_1,\ldots,c_r$, not all zero, and $F(\boldsymbol{y})\in\overline{\mathbb{Q}}(\boldsymbol{y})$ such that

$$\begin{aligned} F(\boldsymbol{y}) &= F(\Omega\boldsymbol{y})+\sum_{k=q}^{p+q-1}\left(\sum_{i=1}^{\rho}b_i\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}\right.\\ &\quad\left.+\sum_{i=1}^{r}\frac{c_i\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}}{1-\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}}\right). \end{aligned} \tag{4}$$

(ii) There are rational integers $d_1,\ldots,d_r$, not all zero, and $G(\boldsymbol{y})\in\overline{\mathbb{Q}}(\boldsymbol{y})\setminus\{0\}$ such that

$$G(\boldsymbol{y})=\left(\prod_{k=q}^{p+q-1}\prod_{i=1}^{r}\left(1-\zeta_i^{a_k}\prod_{j=1}^{s}(y_{j1}^{a_{k+n-1}}\cdots y_{jn}^{a_k})^{e_{ij}}\right)^{d_i}\right)G(\Omega\boldsymbol{y}). \tag{5}$$

Let $M > 0$ be a sufficiently large integer and let

$$F^*(\boldsymbol{z}) = F(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n),$$
$$G^*(\boldsymbol{z}) = G(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n) \setminus \{0\}.$$

Then by (4) and (5), at least one of the following two functional equations holds:

$$\begin{aligned} F^*(\boldsymbol{z}) &= F^*(\Omega \boldsymbol{z}) \\ &+ \sum_{k=q}^{p+q-1} \left(\sum_{i=1}^{\rho} b_i \zeta_i^{a_k} (z_1^{a_{k+n-1}} \cdots z_n^{a_k})^{E_i} + \sum_{i=1}^{r} \frac{c_i \zeta_i^{a_k} (z_1^{a_{k+n-1}} \cdots z_n^{a_k})^{E_i}}{1 - \zeta_i^{a_k} (z_1^{a_{k+n-1}} \cdots z_n^{a_k})^{E_i}} \right), \end{aligned} \tag{6}$$

$$G^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \left(1 - \zeta_i^{a_k} (z_1^{a_{k+n-1}} \cdots z_n^{a_k})^{E_i} \right)^{d_i} \right) G^*(\Omega \boldsymbol{z}), \tag{7}$$

where Ω sends $z_1^{a_{k+n-1}} \cdots z_n^{a_k}$ to $z_1^{a_{k+p+n-1}} \cdots z_n^{a_{k+p}}$ and $E_i = \sum_{j=1}^{s} e_{ij} M^j > 0$ $(1 \le i \le r)$ such that $E_i \neq E_{i'}$ if $\alpha_i/\alpha_{i'}$ is not a root of unity, or equivalently $(e_{i1}, \ldots, e_{is}) \neq (e_{i'1}, \ldots, e_{i's})$. By Theorems 1 and 2 of [7], at least one of the following two properties are satisfied:

(i) For any k $(q \le k \le p + q - 1)$,

$$\sum_{i=1}^{\rho} b_i \zeta_i^{a_k} X^{E_i} + \sum_{i=1}^{r} \frac{c_i \zeta_i^{a_k} X^{E_i}}{1 - \zeta_i^{a_k} X^{E_i}} = \sum_{i=1}^{\rho} b_i \zeta_i^{a_k} X^{E_i} + \sum_{i=1}^{r} c_i \sum_{h=1}^{\infty} (\zeta_i^{a_k} X^{E_i})^h \in \overline{\mathbb{Q}}. \tag{8}$$

(ii) For any k $(q \le k \le p + q - 1)$,

$$\prod_{i=1}^{r} (1 - \zeta_i^{a_k} X^{E_i})^{d_i} = \gamma_k \in \overline{\mathbb{Q}}^{\times}. \tag{9}$$

If (6) is satisfied, then all the coefficients of the right-hand side of (8) must be zero. Therefore, if $c_i = 0$ $(1 \le i \le r)$, then $\alpha_1, \ldots, \alpha_\rho$ are $\{a_k\}_{k \ge 0}$-dependent, which contradicts the assumption. If $c_1, \ldots, c_r$ are not all zero, then $\alpha_1, \ldots, \alpha_r$ are strongly $\{a_k\}_{k \ge 0}$-dependent, which also contradicts the assumption.

If (7) is satisfied, taking the logarithmic derivative of (9), we get

$$\sum_{i=1}^{r} \frac{-d_i E_i \zeta_i^{a_k} X^{E_i - 1}}{1 - \zeta_i^{a_k} X^{E_i}} = 0 \quad (q \le k \le p + q - 1)$$

and so

$$\sum_{i=1}^{r} \frac{d_i E_i \zeta_i^{a_k} X^{E_i}}{1 - \zeta_i^{a_k} X^{E_i}} = \sum_{i=1}^{r} d_i E_i \sum_{h=1}^{\infty} (\zeta_i^{a_k} X^{E_i})^h = 0 \quad (q \leq k \leq p+q-1).$$

Therefore $\alpha_1, \ldots, \alpha_r$ are strongly $\{a_k\}_{k\geq 0}$-dependent in this case by the same way as above. This completes the proof of the theorem.

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